Open problem session
Representation Theory XVI
Dubrovnik June 28, 2019

July 5, 2019

1 Introduction

These are open problems presented at the last meeting of the Lie Groups Section at the conference "Representation Theory XVI" held at the IUC, Dubrovnik, Croatia, June 23–29, 2019. Notes by David Vogan.

2 David Vogan: Geck's conjectural definition of special nilpotent classes

Lusztig in [9] introduced a class of irreducible representations of a Weyl group that he called *special*. All representations of the symmetric group are special, but this is not true for any other type of Weyl group. Lusztig and others proved that special Weyl group representations are deeply entwined with "Kazhdan-Lusztig theory," relating Weyl group representations via Hecke algebras to the representation theory of reductive groups. The Springer correspondence attached to special Weyl group representations certain unipotent classes in G (or, equivalently in the case of $\mathbb{C}$, to nilpotent elements in $\mathfrak{g}^*$), also called *special*. One of the fundamental consequences for representation theory is

Theorem 2.1. *If X is an irreducible representation of a complex reductive Lie algebra $\mathfrak{g}$, then the associated variety of $\mathrm{Ann}(X)$ is the closure in $\mathfrak{g}^*$ of a single special nilpotent coadjoint orbit. Conversely, every special nilpotent coadjoint orbit arises in this way.*

In [4, Conjecture 4.10], Geck defines a simple "integrality condition" on a nilpotent coadjoint orbit which he conjectures is equivalent to Lusztig's notion of special. Here is how. Fix a pinning for $\mathfrak{g}$, meaning a Cartan subalgebra in a Borel subalgebra $\mathfrak{h} \subset \mathfrak{b}$, and a choice of simple root vectors

$$\{X_\alpha \mid \alpha \text{ simple}\}.$$

This defines basis vectors $X_{-\alpha}$ for the negative simple roots by the requirement that
$$[X_\alpha, X_{-\alpha}] = H_\alpha = \text{coroot for } \alpha.$$
From these root vectors we can construct a *Chevalley basis vector* X_γ for each root space; each such vector is well-defined up to sign. The set
$$\{X_\gamma \mid \gamma \in \Delta(\mathfrak{g}, \mathfrak{h})\} \cup \{H_i\}$$
is called a *Chevalley basis* of $\mathfrak{g}$. The structure constants for this basis are integers, so the basis vectors span a natural $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ of $\mathfrak{g}$.

Each nilpotent orbit in $\mathfrak{g}$ has a representative E so that there is an $\mathfrak{sl}(2)$-triple (H, E, F) with
$$\begin{gathered}[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H\\ H \in \mathfrak{h} \text{ dominant.}\end{gathered}$$
The resulting element H is uniquely defined (given the choice of $\mathfrak{h} \subset \mathfrak{b}$) by the orbit of E. Necessarily H is a nonnegative integer combination of the simple coroots H_α, so
$$\gamma(H) \in \mathbb{N} \qquad (\gamma \in \Delta(\mathfrak{g}, \mathfrak{h})).$$
This means that the eigenspaces of H define a $\mathbb{Z}$-grading
$$\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i.$$
Each $\mathfrak{g}_i$ for $i \neq 0$ is spanned by roots, and therefore defined over $\mathbb{Z}$; and $\mathfrak{g}_0$ is spanned by roots and $\mathfrak{h}$, and therefore also defined over $\mathbb{Z}$. Necessarily
$$E \in \mathfrak{g}_2, \quad H \in \mathfrak{g}_0, \quad F \in \mathfrak{g}_{-2}.$$
Every linear functional $\epsilon' \in \mathfrak{g}_2^*$ defines a skew-symmetric bilinear form on $\mathfrak{g}_1$ by
$$\omega_{\epsilon'}(x, y) =_{\text{def}} \epsilon'([x, y]).$$
If $\epsilon' \in \mathfrak{g}_{2,\mathbb{Z}}^*$ (that is, if ϵ' takes integer values on the Chevalley basis vectors $\{X_\gamma\}$), then $\omega_{\epsilon'}$ is defined over $\mathbb{Z}$; that is, we get an integer matrix
$$\omega_{\epsilon'}(X_{\gamma_i}, X_{\gamma_j}) \in \mathbb{Z} \qquad (\gamma_i(H) = \gamma_j(H) = 1)$$
of size the dimension of $\mathfrak{g}_1$. If B is a nondegenerate invariant bilinear form on $\mathfrak{g}$ and
$$\epsilon(z) =_{\text{def}} B(F, z) \qquad (z \in \mathfrak{g}_2),$$
then the Kirillov-Kostant theory of coadjoint orbits guarantees that the symplectic form ω_ϵ is nondegenerate. It follows that for "most" integral ϵ', the form $\omega_{\epsilon'}$ has nonzero (integral) determinant. Geck's conjecture is

$$\begin{gathered}E \text{ is special if and only if there is an integral } \epsilon' \text{ so that}\\ \det(\omega_{\epsilon'}(X_{\gamma_i}, X_{\gamma_j}) = \pm 1 \qquad (\gamma_i(H) = 1).\end{gathered}$$
Geck's conjecture suggests three problems.

1. Show that if X is a simple $\mathfrak{g}$-module defined over $\mathbb{Z}$, then X has integral infinitesimal character.

2. Show that I is a primitive ideal of integral infinitesimal character, then $I = \mathrm{Ann}(X)$ for some simple $\mathfrak{g}$-module defined over $\mathbb{Z}$.

3. Show that if X is a simple Harish-Chandra module defined over Z, then some representative $\lambda \in \mathfrak{g}^*_{\mathbb{Z}}$ of an open orbit in the associated variety $\mathrm{AV}(X)$ must satisfy Geck's integrality condition.

Perhaps (1) is not too difficult. Part (2) is probably immediate from Duflo's theorem relating primitive ideals to highest weight modules. Part (3) is meant to be analogous to Gabber's "integrability of characteristic" theorem [3]; it may be difficult.

Here are two related problems.

4. A linear functional ϵ' on $\mathfrak{g}_2$ as above, extended by zero on all other $\mathfrak{g}_1$, defines a nilpotent coadjoint orbit $G \cdot \epsilon'$, which carries a Kirillov-Kostant symplectic structure $\Omega_{\epsilon'}$. Show that this structure is naturally defined over $\mathbb{Z}$ if and only if ϵ' takes integer values on the Chevalley basis; and in this case the structure can be chosen nondegenerate over $\mathbb{Z}$.

5. Study $(\mathfrak{g}, K)$-modules which are defined over $\mathbb{Z}$. (One possible guess is that the irreducibles defined over $\mathbb{Z}$ are precisely those in the block of finite-dimensional representations.)

3 Dan Ciubotaru: counting elliptic elements of Weyl groups

Suppose $W = W(\mathfrak{g}, \mathfrak{h})$ is the Weyl group of a semisimple Lie algebra $\mathfrak{g}$. An element $w \in W$ is called *elliptic* if it does not have the eigenvalue 1 on $\mathfrak{h}$. Define a class function on W

$$\mathbb{1}_{\mathrm{ell}}(w) = \begin{cases} 1 & w \text{ elliptic} \\ 0 & w \text{ not elliptic.} \end{cases}$$

Suppose now that H, E, F is a Lie triple for a distinguished nilpotent element (meaning that the centralizer in $\mathfrak{g}$ of (H, E, F) is the center of $\mathfrak{g}$). Then the conjecture is

$$\langle \mathbb{1}_{\mathrm{ell}}, H^\bullet(\mathcal{B}_E) \rangle_W = \frac{\prod'_{\alpha \in R(\mathfrak{g},\mathfrak{h})} \alpha(H)}{\prod'_{\alpha \in R(\mathfrak{g},\mathfrak{h})} (\alpha(H) - 2)}.$$

Here each product runs over *all* roots of $\mathfrak{h}$ in $\mathfrak{g}$; the prime means that factors equal to zero are to be omitted. The Springer fiber $\mathcal{B}_E$ consists of all Borel subalgebras containing E.

Suppose for example that E is a principal nilpotent element. Then $\mathcal{B}_E$ is a single point (carrying the trivial representation of W), so the left side of the conjecture is

$$\langle \mathbb{1}_{\mathrm{ell}}, \mathrm{trivial}\rangle_W = |W_{\mathrm{ell}}|/|W|.$$

The right side of the conjecture is computed by Kostant's description of the decomposition of $\mathfrak{g}$ under a principal three-dimensional subalgebra; it is

$$\prod_i \frac{m_i}{m_i+1}.$$

Write $n = \dim \mathfrak{h}$ and

$$W_j = \{w \in W \mid \dim \mathfrak{h}^w = n - j\},$$

the elements of W for which the eigenvalue 1 has multiplicity $n-j$. A formula due to Shephard and Todd says that

$$\sum_j t^j |W_j| = \prod_i (1 + m_i t),$$

with m_i the exponents of W. Consequences include

$$|W| = \prod_i (m_i + 1), \qquad |W_{\mathrm{ell}}| = \prod_i m_i.$$

It follows that the conjecture is true for the principal nilpotent element.

4 David Renard: resolutions for character formulas for p-adic group representations

Suppose that $G_n = GL(n, F)$, with F a p-adic field. Zelevinski in 1980 gave a classification of the irreducible representations of G_n in terms of the supercuspidal representations of all $G_{n'}$ (for $n' \leq n$) and some combinatorial data called *multisegments*. One can find a clear account for example in [10]. A *segment* is a sequence of integers increasing by 1

$$\Delta = \{b, b+1, \dots, e\} =_{\mathrm{def}} [b, e] \qquad (b \leq e \in \mathbb{Z})$$

The *length* $\ell(\Delta)$ of the segment is its cardinality $e - b + 1$. A *multisegment* is a finite multiset of segments

$$\underline{m} = (\Delta_1, \dots, \Delta_t),$$

unordered but counted with multiplicity. The *length* $\ell(\underline{m})$ of the multisegment is the sum (with multiplicities) of the lengths of its constituent segments. For example,

$$\ell(\{-4,-3,-2\}, \{-4,-3,-2\}, \{-3,-2,-1,0,1\}) = 11.$$

Using a pair $(\rho, \underline{m})$ (consisting of a supercuspidal ρ of G_d and a multisegment of length ℓ) Zelevinsky defines a standard representation $\mathrm{std}(\underline{m}, \rho)$ of $G_{d\ell}$ having a unique irreducible quotient $\mathrm{irr}(\underline{m}, \rho)$. Zelevinski proves that this construction parametrizes irreducibles of $G_{d\ell}$ having supercuspidal support ρ by multisegments $\underline{m}$ of length ℓ. (Other irreducible representations of G_n are obtained from these basic ones by irreducible parabolic induction.)

For integers $b \le e$, the representation $\mathrm{irr}([b,e],\rho) = \mathrm{std}([b,e],\rho)$ is an essentially square integrable modulo the center representation of $G_{(b-e+1)d}$. It is convenient to define formally $\mathrm{irr}([b,b-1],\rho)$ to be the one-dimensional trivial representation of the trivial group G_0, and $\mathrm{irr}(\rho,[b,e]) = 0$ if $e \le b-2$.

If π_1, ...,π_r are representation of G_{n_1}, ..., G_{n_r} respectively, we write as usual

$$\pi_1 \times \cdots \times \pi_r$$

for the representation of $G_n = G_{n_1+\cdots+n_r}$ parabolically induced from the standard parabolic subgroup of G_n of type $(n_1,\ldots,n_r)$ (with Levi subgroup $G_{n_1} \times \cdots \times G_{n_r}$.

We say that the segment $[b,e]$ *precedes* $[b',e']$ if $b < b' \le e+1 < e'+1$.

If $\underline{m} = ([b_1,e_1]),\ldots,[b_t,e_t])$ is a multisegment such that $[b_i,e_i]$ does *not* precede $[b_j,e_j]$ for $i<j$, we say that the multisegment is presented in a *standard order*, and then the representation

$$\mathrm{std}(\underline{m},\rho) = \mathrm{irr}(\rho,[b_1,e_1) \times \cdots \times \mathrm{irr}(\rho,[b_t,e_t]$$

has a unique irreducible quotient $\mathrm{irr}(\underline{m},\rho)$. Of course, we can always present a multisegment in a standard order, and $\mathrm{std}(m,\rho)$, $\mathrm{irr}(m,\rho)$ do not depend on the chosen standard order.

The standard representations $\mathrm{std}(\underline{m},\rho)$ are rather completely understandable in terms of ρ. In order to understand irreducible representations, it is therefore of interest to express the irreducible representations in terms of standard ones:

$$\mathrm{irr}(\underline{m},\rho) = \sum_{\underline{m}'} a(\underline{m}',\underline{m})\,\mathrm{std}(\underline{m}',\rho).$$

This is accomplished by Kazhdan-Lusztig theory: the integers $a(\underline{m}',\underline{m})$ are given by values at $q=1$ of certain Kazhdan-Lusztig polynomials (which are depend not on the supercuspidal ρ but only on the combinatorics of multisegments). This was conjectured by Zelevinski [16], [17], and proved by Chriss and Ginzburg [1].

A *Speh representation* corresponds to a multisegment

$$\underline{m}_{\mathrm{Speh}} = (\{b,b+1,\ldots,e\},\{b+1,\ldots,e+1\},\ldots,\{b+p-1,\ldots,e+p-1\})$$

consisting of p segments of the same length, each shifted one step to the right of its predecessor. The symmetric group S_p "acts" on this multisegment: for each $\tau \in S_p$, $\tau \cdot \underline{m}$ is another multisegment with the same support as τ. This is *not* a group action. We will not give the definition of $\tau \cdot \underline{m}$ in general, but here are some examples with $p = 3$. First, the identity element of S_p always acts trivially. Second,

$$\sigma_{12} \cdot (\{1\}, \{2\}, \{3\}) = (\{1,2\}, \{3\}),$$
$$\sigma_{12} \cdot (\{1,2,3\}, \{2,3,4\}, \{3,4,5\}) = (\{1,2,3,4\}, \{2,3\}, \{3,4,5\})$$

$$\sigma_{23} \cdot (\{1\}, \{2\}, \{3\}) = (\{1\}, \{2,3\}),$$
$$\sigma_{23} \cdot (\{1,2,3\}, \{2,3,4\}, \{3,4,5\}) = (\{1,2,3\}, \{2,3,4,5\}, 3, 4)$$

$$\sigma_{123} \cdot (\{1\}, \{2\}, \{3\}) = (\{1,2,3\}),$$
$$\sigma_{123} \cdot (\{1,2,3\}, \{2,3,4\}, \{3,4,5\}) = (\{1,2,3,4\}, \{2,3\}, \{3,4,5\})$$

$$\sigma_{321} \cdot (\{1\}, \{2\}, \{3\}) = (\{1,2,3\}),$$
$$\sigma_{321} \cdot (\{1,2,3\}, \{2,3,4\}, \{3,4,5\}) = (\{1,2,3\}, \{3,4\}, \{2,3,4,5\}).$$

When these formulas are appropriately corrected and generalized, Tadić proved in [13] that

$$\mathrm{irr}(\underline{m}_{\mathrm{Speh}}, \rho) = \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma)\, \mathrm{std}(\sigma \cdot \underline{m}_{\mathrm{Speh}}, \rho).$$

Lapid and Mínguez in [8] generalized this formula to "ladder" multisegments: those of the form

$$\underline{m}_{\mathrm{ladder}} = ([b,e], [b + x_1, e + y_1], \ldots, [b + x_{p-1}, e + y_{p-1}]),$$
$$0 < x_1 < \cdots < x_{p-1}, \qquad 0 < y_1 < \cdots < y_{p-1}.$$

The problem proposed by Renard is to find a proof of this Lapid-Mínguez character formula along the lines of the BGG resolution: by constructing a resolution

$$0 \leftarrow \mathrm{irr}(\underline{m}_{\mathrm{ladder}}, \rho) \leftarrow \mathrm{std}(\underline{m}_{\mathrm{ladder}}, \rho) \leftarrow \sum_{\ell(\sigma)=1} \mathrm{std}(\sigma \cdot \underline{m}_{\mathrm{ladder}}, \rho) \leftarrow \cdots .$$

Lapid and Mínguez construct the first one or two terms of such a resolution, then use a series of tricks to deduce the character formula.

Renard also asks: for real reductive groups, when are there resolutions of irreducible Harish-Chandra modules giving rise to simple character formulas? Using Beilinson-Bernstein localisation, such resolutions can obtained from resolutions called *Cousin complexes* in a category of $\mathcal{D}_\lambda$-modules. Dragan Milicic has a set of notes about these.

5 Jeff Adams: extending `atlas`

The `atlas` software [15] does a wide variety of calculations related to the structure and representation theory of $G(\mathbb{R})$, with G a complex connected reductive algebraic group. Extend the software to treat (finite) nonlinear coverings of such a group. It may be necessary to find a good class of such nonlinear groups. One interesting example is all two-fold covers of $G(\mathbb{R})$.

David Vogan suggested along similar lines that one could try to extend either the Langlands program, or `atlas`, or $\mathcal{D}$-modules, to cover representations of the real points of possibly *disconnected* complex reductive algebraic groups. The most familiar group of this sort is $O(n)$. Many others appear in applications, for example as centralizers or normalizers of reductive subgroups of connected reductive algebraic groups.

6 David Vogan: dual pairs

Suppose G is a group. A *dual pair* in G is a pair of subgroups H_1 and H_2 with the property that

$$H_2 = \mathrm{Cent}_G(H_1), \qquad H_1 = \mathrm{Cent}_G(H_2).$$

If G is algebraic, then H_1 and H_2 are automatically algebraic as well. The problem is to classify dual pairs up to conjugation in G (perhaps for G complex reductive algebraic).

Roger Howe's theory of dual pairs concerns the case $G = Sp(2n)$ and H_i reductive; he classified such pairs completely in [5]. His classification was extended by Rubenthaler [12] to dual pairs of complex reductive Lie algebras in a complex reductive $\mathfrak{g}$.

For a rather different example, suppose G is a simple complex group of type G_2, H_1 is the $PSL(2)$ subgroup corresponding to a subregular orbit, and $H_2 = \mathrm{Cent}_G(H_1)$. Then H_2 is the symmetric group on three letters, and (H_1, H_2) is a dual pair.

If H_0 is any subgroup of any group G, and

$$H_1 = \mathrm{Cent}_G(H_0), \qquad H_2 = \mathrm{Cent}_G(H_1),$$

then (H_1, H_2) is a dual pair.

7 Kyo Nishiyama: reducible associated varieties

Suppose that X is an irreducible $(\mathfrak{g}, K)$-module (with G a complex connected reductive algebraic group and $K = G^\theta$ the fixed points of an involutive automorphism). Then the associated variety of X is decomposed into irreducible components

$$\mathrm{AV}(X) = \cup_{i=1}^r \overline{\mathcal{O}_i};$$

the various $\mathcal{O}_i$ are K-invariant Lagrangians in a common nilpotent coadjoint orbit $\mathcal{O}$. (Here $\mathcal{N}^* \subset \mathfrak{g}^*$ is the nilpotent cone and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ the Cartan decomposition defined by θ.) The conjecture is that "$\mathrm{AV}(X)$ is connected in codimension 1." This means that if $\mathcal{O}_i$ and $\mathcal{O}_{i'}$ are two of these components, then they are connected by a chain of components

$$\mathcal{O}_i = \mathcal{O}_{j_1},\ \mathcal{O}_{j_2},\ \cdots,\ \mathcal{O}_{j_r} = \mathcal{O}_{i'},$$

so that

$$\overline{\mathcal{O}_{j_k}} \cap \overline{\mathcal{O}_{j_{k+1}}} \ \text{ has codimension one in each,} \qquad 1 \le k \le r-1.$$

This conjecture seeks to sharpen a result of [14], which says that if $\mathrm{AV}(X)$ is reducible, then each irreducible component $\overline{\mathcal{O}}_i$ must have codimension 1 boundary.

Lucas Mason-Brown has observed that the "Hodge filtration conjecture" of Schmid and Vilonen (that the Hodge filtration sheaves for an irreducible $\mathcal{D}_\lambda$-module attached to X have no higher cohomology) would imply that the Hodge filtration makes $\mathrm{gr}(X)$ Cohen-Macaulay, and therefore that the associated variety is connected in codimension one ([2, page 454]).

In the case $G = GL(n)$, $K = GL(p) \times GL(q)$, it is proven in [11] that every "codimension one connected component" is in fact an associated variety of an irreducible $(\mathfrak{g}, K)$-module.

Kashiwara has proven the analogous codimension one connectedness statement for characteristic cycles of irreducible regular holonomic $\mathcal{D}$-modules ([6], [7, Theorem 1.2.2]). (This result is also a consequence of Saito's Hodge filtration.)

References

[1] Neil and Ginzburg Chriss Victor, *Representation theory and complex geometry*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1997 edition.

[2] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[3] O. Gabber, *The integrability of the characteristic variety*, Amer. J. Math. **103** (1981), 445–468.

[4] Meinolf Geck, *Generalised Gelfand-Graev representations in bad characteristic?* (2018), 22 pp., available at `arXiv:1810.08937v2[math.RT]`.

[5] R. Howe, *θ-series and invariant theory*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 275–285.

[6] Masaki Kashiwara, *Introduction to microlocal analysis*, Enseign. Math. (2) **32** (1986), no. 3-4, 227–259.

[7] Masaki Kashiwara and Takahiro Kawai, *On holonomic systems of microdifferential equations. III. Systems with regular singularities*, Publ. Res. Inst. Math. Sci. **17** (1981), no. 3, 813–979.

[8] Erez Lapid and Alberto Mínguez, *On a determinantal formula of Tadić*, Amer. J. Math. **136** (2014), no. 1, 111–142.

[9] G. Lusztig, *A class of irreducible representations of a Weyl group*, Proc. Kon. Nederl. Akad., A **82** (1982), 323–335.

[10] C. Mœglin and J.-L. Waldspurger, *Sur l'involution de Zelevinski*, J. Reine Angew. Math. **372** (1986), 136–177 (French).

[11] Kyo Nishiyama, Peter Trapa, and Akihito Wachi, *Codimension one connectedness of the graph of associated varieties*, Tohoku Math. J. (2) **68** (2016), no. 2, 199–239.

[12] Hubert Rubenthaler, *Les paires duales dans les algèbres de Lie réductives*, Astérisque **219** (1994), 121 (French, with English and French summaries).

[13] M. Tadić, *On characters of irreducible unitary representations of general linear groups*, Abh. Math. Sem. Univ. Hamburg **65** (1995), 341–363.

[14] David A. Vogan Jr., *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., vol. 101, Birkhäuser Boston, Boston, MA, 1991, pp. 315–388.

[15] *Atlas of Lie Groups and Representations software*, 2019. `http://www.liegroups.org`.

[16] A. V. Zelevinskiĭ, *The p-adic analogue of the Kazhdan-Lusztig conjecture*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 2, 9–21, 96 (Russian); English transl., Functional Anal. Appl. **15**, no. 2, 83-92.

[17] ______, *Two remarks on graded nilpotent classes*, Uspekhi Mat. Nauk **40** (1985), no. 1(241), 199–200 (Russian); English transl., Russian Math. Surveys **40**, no. 1, 249-250.